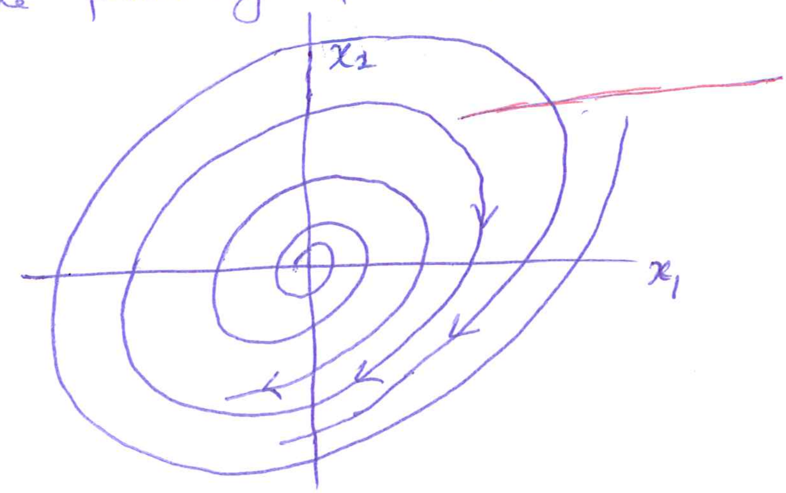
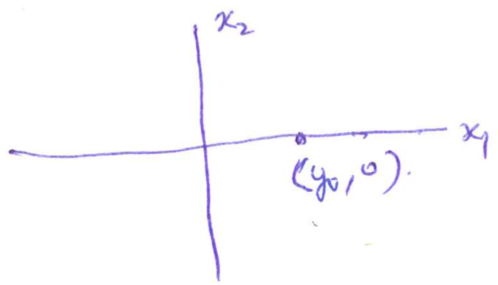
STABILITY OF LIMIT CYCLES, POINCARE MAPS, FINDING LIMIT CYCLES, etc

Consider the following phase portrait with a stable limit cycle.



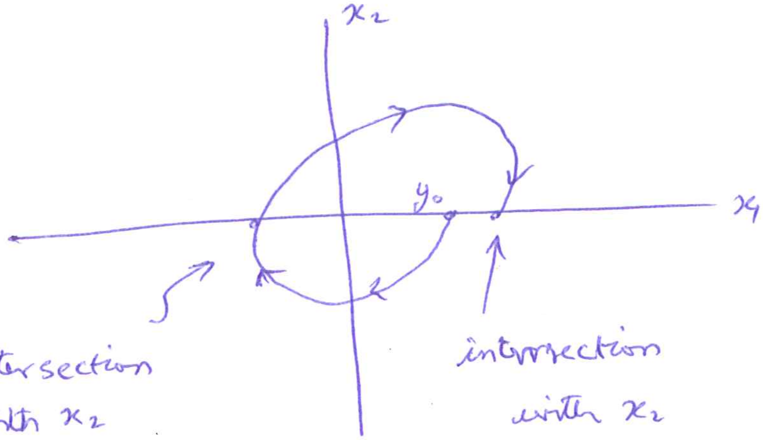
Say we start the simulation at some initial condition of the form $(y_0, 0)$ with $y_0 > 0$.



As we run the ODE forward in time the trajectory spirals around the origin and intersects the $x_2 = 0$ (horizontal) axis again and again and again.

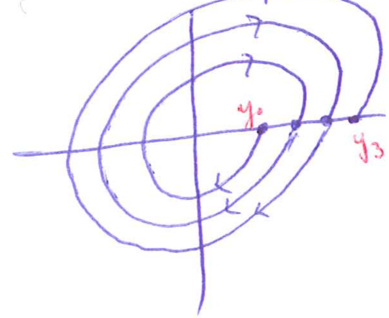
Let us pay attention to where the trajectory intersects the $x_2 = 0$ axis as x_2 is decreasing.

We ignore intersections of the trajectory with the $x_2 = 0$ axis if x_2 is increasing across the intersection



intersection with x_2 increasing $\dot{x}_2 > 0$ — ignore

intersection with x_2 decreasing $\dot{x}_2 < 0$. — call this $(y_1, 0)$.



Next intersection $(y_2, 0)$ say. More generally, the i-th intersection is called y_i.

One can imagine a function taking as input y_i and giving y_{i+1} as output.

Such a function is called a "Poincaré map" or a "first return map" or simply a "return map".

By focusing on the relation between successive intersections with the $x_2 = 0$ axis, we are trying to reduce the original continuous time ODE to a discrete dynamical system.

The line $x_2 = 0$ with $\dot{x}_2 > 0$ is called a Poincaré section.

We can represent the Poincaré map one of two ways

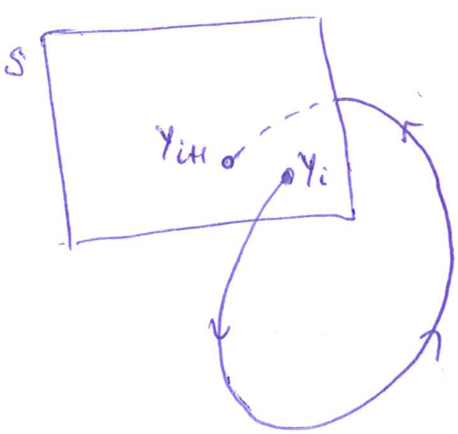
$$\begin{bmatrix} y_{i+1} \\ 0 \end{bmatrix} = P\left(\begin{bmatrix} y_i \\ 0 \end{bmatrix}\right).$$

(or) $\quad y_{i+1} = P(y_i)$. $\Leftarrow$ ignoring the mapping of 0 to 0 by definition of the Poincaré map.

We will adapt the latter convention. This convention means that for a N-dimensional first order ODE $\dot{x} = f(x)$, the Poincare map will be

$Y_{i+1} = P(Y_i)$ where Y is a (N-1)-dimensional vector.

In many dimensions; the surface of section - the Poincaré section is N-1 dimensional.



The surface of section, the Poincaré section is required to be "<u>transversal</u>" (that is, never tangential) to the trajectories.

The surface Poincare section <u>need</u> <u>not</u> be flat like a line or a plane but can be a curve or a curved surface.

In general, the Poincaré map P is a function that takes points on the surface S to points on the surface S, by following the trajectories of the ODE from one intersection with S to the next intersection.

Note: The actual function $P(Y)$ is hard to compute in closed-form for any system of even minimal complexity. So almost always, the Poincaré map is computed by numerical integration of the ODE.

- Given some initial condition on the Poincaré section, we integrate forward in time until the trajectory intersects the Poincaré section again.

- MATLAB's ode45 has "event detection" functionality which is helpful to very accurately stop the integration when the event — such as intersection with the Poincaré section — occurs.

Using MATLAB's event detection requires the definition of a separate function defining the event.

eg. function [value, direction, isterminal] = eventsfile (t, X)

```
    value = X(2);        % stops when X(2) = 0.
    direction = -1;      % stops only when "value" decrease
    isterminal = 1;      % stops the integration
                           as opposed to just
                           noting the event and
                           continuing the
                           integration.
```

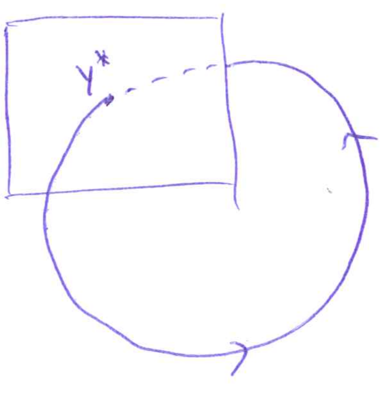
In the file calling the ode45, the options must have defined
options = odeset('Events', @eventsfile).

Now, the poincaré map will be a MATLAB function that takes as input initial conditions Y on the Poincaré section and outputs the state of the system Y on the next intersection with the Poincaré section., obtained by ODE integration and event detection.

LIMIT CYCLES.

Given a Poincaré map $Y_{i+1} = P(Y_i)$, a fixed point Y^* of the Poincaré map corresponds to a point on the limit cycle.

Because $Y^* = P(Y^*)$ i.e. $Y_{i+1} = Y^*$ if $Y_i = Y^*$.



$\Rightarrow Y_i = Y^*$ for all i

$\Rightarrow$ The trajectory is a closed curve that repeats itself over and over again – which is a limit cycle.

Thus, to find an initial condition on a limit cycle we will have to solve the nonlinear simultaneous equations $Y = P(Y)$ – in other words, find the fixed point of P.

This can be done by using fsolve in MATLAB for instance, as will be noted later.

STABILITY OF LIMIT CYCLES

Because we have reduced the continuous ODE to a discrete dynamical system, and we have reduced a limit cycle to a fixed point of the discrete dynamical system (namely the Poincaré map) — the limit cycle's stability can be examined by studying the stability of the FP of the Poincaré map P.

This will usually need to be done by computing the Jacobian J of the Poincaré map P, usually by using numerical methods, for differentiation as will be discussed later.

One then computes the eigenvalues of the Jacobian J and checks whether they are all within the unit circle $(\|\lambda_j\| < 1)$ or not.

The eigenvalues of the linearization of the Poincaré map are also sometimes called the <u>characteristic multipliers</u> or the <u>Floquet multipliers</u>.